128 KSME Journal, Vol. 1, No. 2, pp. 128~132, 1987.

SHAPE DESIGN SENSITIVITY ANALYSIS BASED ON BOUNDARY INTEGRAL EQUATION METHOD CONSIDERING GENERAL SHAPE VARIATIONS

PART II : FOR POTENTIAL, ELASTICITY AND PLATE BENDING PROBLEMS

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(Received April 10, 1987)

This work is based upon the general formula for the shape design sensitivity of the elliptical operator. The classical engineering problems of potential, elasticity, and plate bending will be addressed. The derived formulas are suitable for computational purposes in conjunction with the boundary element method. Specific individual problems serve as illustration and their results are compared with those obtained by a different approach which is based on a variational method over the domain.

Key Words: Sensitivity Analysis, Body Force, Shape Design, Plane Elasticity

1. INTRODUCTION

A formula for the sensitivity of functionals expressed in domain and boundary integrals has been derived for general elliptic opertor equations, based upon a boundary integral equation formulation. (Kwak, 1987a) Because the formula is written in general terms, considerable manipulation is necessary to obtain concrete formulas for specific application problems. The purpose of this paper is thus to present specialized concrete expressions of sensitivity as applied to typical elliptical engineering problems such as potential, elasticity and plate bending problems. In regard to elasticity, the axisymmetric problem will be considered here, because the plane elasticity problem has already been discussed with some specific examples of a fillet and an elastic ring problem. (Kwak, 1987b) Most of the following derivations involve operations in material derivatives and integration by parts. (Zolesio, 1981).

2. POTENTIAL PROBLEM

Consider a potential u defined on an arbitrary domain Ω as shown in Fig. 1. For simplicity, the derivation is restricted to two-dimensional problems. The formal boundary value problem for u can be written by the Laplacian operator, order $2m=2$,

$$Au=-\nabla^2 u=f(x), \qquad x\in\Omega, \tag{1}$$

$$\left.\begin{array}{ll} u=b, & x\in\partial\Omega_0, \\ p(u)\equiv\dfrac{\partial u}{\partial n}=c, & x\in\partial\Omega_1, \end{array}\right\} \tag{2}$$

whrere f, b and c are the prescribed functions, and $p(u)$ is the flux of u on the boundary. The boundary operators B_0 and C_0 appearing in Kwak's general formulation (Kwak, 1987a) are represented here by 1 and $\partial/\partial n$. The corresponding integral identity is

$$\int_{\partial\Omega}\{up(w)-p(u)w\}ds=\int_\Omega(fw-gu)\,dx, \tag{3}$$

where w is an arbitrary potential satisfying (1) with f replaced by g. A general functional for this problem is now written as

$$\Phi=\int_\Omega h(u,\ \nabla u)\,dx+\int_{\partial\Omega}\Psi\{u,\ p(u)\}ds. \tag{4}$$

Noting that

$$\int_\Omega h_{\nabla u}\cdot\nabla\dot{u}dx=\int_{\partial\Omega}(h_{\nabla u}\cdot n)\,\dot{u}ds-\int_\Omega(div\ h_{\nabla u})\,\dot{u}\,dx, \tag{5}$$

where $h_{\nabla u}$ denototes the partial derivative with respect to ∇u, the material derivative of Φ becomes

$$\begin{aligned}\Phi'=&\int_\Omega(h_u-div\ h_{\nabla u})\,\dot{u}\,dx+\int_{\partial\Omega}\{(\Psi_u+h_{\nabla u}\cdot n)\,\dot{u}\\&+\Psi_p\dot{p}\}ds-\int_\Omega\{h_u\nabla u\cdot V+h_{\nabla u}\cdot\nabla(\nabla u\cdot V)\}dx\\&+\int_{\partial\Omega}\{(h+\Psi H)\,V_n+\Psi V_{s,s}\}ds.\end{aligned} \tag{6}$$

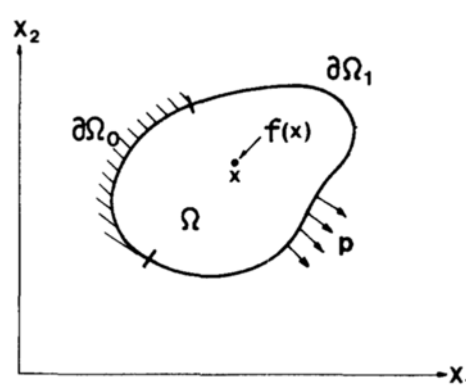

Fig. 1 Potential problem

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where V_s is the tangential component of the velocity field V. For a two dimensional case, the vector V_T defined by Kwak (Kwak, 1987a) becomes simply $V_s s$, where s denotes the unit tangential vector on $\partial\Omega$. Note in (4) that the function h contains another argument ∇u, which was not considered in the general procedure (Kwak, 1987a). However, as noted by Kwak (Kwak, 1987a), using relation (5), the material derivative of ∇u has been eliminated, resulting in expression (6).

In order to relate $\dot{u}$ and $\dot{p}$ in (6) with the velocity field V, the material derivative of identity (3) is taken, noting the following identity.

$$p(\dot{w}) = p(w') + p_{,n}(w)\, V_n + p_{,s}(w)\, V_s - w_{,s} V_{n,s} \tag{7}$$

Therefore, the operators β_0 and γ_0 in the general sensitivity formula in Kwak (Kwak, 1987a), are identified as follows:

$$\begin{aligned} \beta_0(V) &= V_n \frac{\partial}{\partial n} + V_s \frac{\partial}{\partial s}, \\ \gamma_0(V) &= V_n \frac{\partial^2}{\partial n^2} + V_s \frac{\partial^2}{\partial s \partial n} - V_{n,s} \frac{\partial}{\partial s} \end{aligned} \tag{8}$$

The next manipulation is to take the material derivative of (3) and to substitute (7) into the resulting expression. The following indentities are also used:

$$p_{,n}(w) + p(w) H + w_{,ss} \equiv \nabla^2 w = -g, \tag{9}$$

and $$\int_{\partial\Omega} \{u p(w') - p(u) w'\} ds = \int_\Omega (fw' - g' u)\, dx \tag{10}$$

Using (7), (9) and (10), and after some simplifications, one obtains the expression for the variation of (3),

$$\begin{aligned} &\int_\Omega g \dot{u}\, dx + \int_{\partial\Omega} \{p(w)\, \dot{u} - w \dot{p}(u)\} ds = \int_\Omega g \nabla u \cdot V\, dx \\ &+ \int_{\partial\Omega} [\{fw - u_{,s} w_{,s} + p(u)\,(p(w) + wH)\} V_n \\ &+ \{u_{,s} p(w) - p_{,s}(u)\, w\} V_s] ds. \end{aligned} \tag{11}$$

To relate (11) with (6), introduce an adjoint system such that

$$f^* = h_u - div\, h_{\nabla u} \quad in\ \Omega, \tag{12}$$

$$\begin{aligned} u^* &= -\Psi_p && on\ \partial\Omega_0, \\ p^* &= \Psi_u + h_{\nabla u} \cdot n && on\ \partial\Omega_1. \end{aligned} \tag{13}$$

Substituting u^* and f^* in place of w and g in (11) and utilizing (12) and (13), one finally obtains the desired sensitivity formula for Φ as

$$\begin{aligned} \Phi' = &\int_{\partial\Omega} [\{h + \Psi H + fu^* - u_{,s} u^*_{,s} + p(p^* - h_{\nabla u} \cdot n \\ &+ u^* H)\} V_n + \{(u_{,s}(p^* - h_{\nabla u} \cdot n) - p_{,s} u^*)\, V_s \\ &+ \Psi V_{s,s}\}] ds + \int_{\partial\Omega_0} (\Psi_u + h_{\nabla u} \cdot n - p^*)\, \dot{b}\; ds + \int_{\partial\Omega_1} \\ &(\Psi_p + u^*)\, \dot{c}\; ds. \end{aligned} \tag{14}$$

A membrane deflection problem is taken for illustration as a specific case. Consider the membrane shown in Fig. 2, with uniform tension of unit value, and an applied lateral load, f. Under the condition that the deflection u is zero on the boundary, the formal boundary value problem is

$$\begin{aligned} -\nabla^2 u &= f && in\ \Omega, \\ u &= 0 && on\ \partial\Omega. \end{aligned} \tag{15}$$

Consider a functional that represents strain energy of the membrane,

$$\Phi = \frac{1}{2} \int_\Omega \nabla u \cdot \nabla u\, dx. \tag{16}$$

Then, h and Ψ in (4) become $\frac{1}{2} \nabla u \cdot \nabla u$ and zero. Noting from (12) and (13) that the adjoint variable u^* is identical to u, the sensitivity formula (14) becomes simply

$$\Phi' = \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial u}{\partial n} \right)^2 V_n\, ds. \tag{17}$$

The same result is obtained, using a variational formulation, specifically applicable to domain integrals only. (Choi, 1983)

3. AXISYMMETRIC ELASTICITY PROBLEM

Treatment of an axisymmetric elasticity problem differs from the usual procedure described earlier, in that it involves vector functions in (r, z) coordinates. However, the general principle is the same as that for problems of potential.

Consider a linear elasticity problem for an axisymmetric body Ω with smooth boundary, $\partial\Omega$, as shown in Fig. 3. For a cylindrical coordinate system with point x denoted by (r, z), the boundary integral equation can be written as

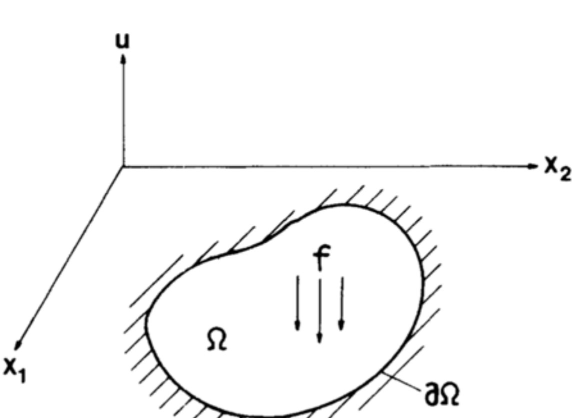

Fig. 2 Membrane deflection with uniform tension

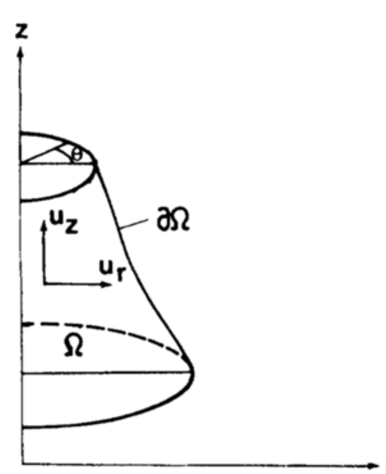

Fig. 3 Axisymmetric elasticity problem

$$\alpha u_i(x_0) + \int_{\partial\Omega} \{u_k(x) F_{ki}(x, x_0) - T_k(x) G_{ki}(x, x_0)\} r ds = \int_\Omega f_k(x) G_{ki}(x, x_0) r\, dx, \quad \text{with } \alpha = \begin{cases} 1, & x_0 \varepsilon \Omega, \\ \frac{1}{2}, & x_0 \varepsilon \partial\Omega, \end{cases} \tag{18}$$

where ds and dx denote the line element of the boundary $\partial\Omega$ and area element of the domain Ω at point x, respectively. The tensor notations are used here with indices representing either direction r or z. The kernal functions $G_{ki}(x, x_0)$ and $F_{ki}(x, x_0)$ are the displacements and the tractions, respectively, in the direcion k at x, due to a unit ring load in the i-th direction applied at x_0. The detailed expressions can be found in "Banerjee, 1981". The function, f_k, appearing in the domain integral denotes rotational body forces, such as centrifugal loading. Let the boundary conditions be written as follows

$$\begin{aligned} u_i &= b_i && \text{on } \partial\Omega_0, \\ T_i &= c_i && \text{on } \partial\Omega_1, \end{aligned} \tag{19}$$

where b_i and c_i are the prescribed displacements and tractions, respectively. Substituting (19) into (18) and solving the resulting integral equations, the unknown terms on the boundary are determined.

As in the potential problem, introducing an arbitrary system of displacements w_i with body forces g_i, the foregoing BIE can be transformed to the following integral identity

$$\int_{\partial\Omega} \{u_i T_i(w) - t_i(u) w_i\} r\, ds = \int_\Omega (f_i w_i - g_i u_i) r\, dx. \tag{20}$$

This result corresponds to Betti's reciprocal theorem for two arbitrary equilibrium states with u_i, f_i and w_i, g_i in only in an axisymmetric body. This equation differs from the plane elasticity case that the radius r appears in the integrals of (20).

Consider a general functional for axisymmetric elasticity problems in the following form:

$$\Phi = \int_\Omega h(u_i) r\, dx + \int_{\partial\Omega} \Psi(u_i, T_i) r\, ds, \tag{21}$$

Note here that r has to be included, such that it is consistent with the BIE formulation. The material derivative of Φ is obtained,

$$\begin{aligned} \Phi' = {} & \int_\Omega h_{ui}\, \dot{u}_i r\, dx + \int_{\partial\Omega} \{\Psi_{ui}\, \dot{u}_i + \Psi_{Ti}\, \dot{T}_i\} r\, ds \\ & - \int_\Omega h_{ui}\, r \nabla u_i \cdot V\, dx + \int_{\partial\Omega} [\{(h + \Psi H) V_n \\ & + \Psi V_{s,s}\} r + \Psi (n_r V_n - n_z V_s)] ds, \end{aligned} \tag{22}$$

where n_r and n_z denote the components of the unit normal on $\partial\Omega$.

Following the general procedure, the material derivative of the identity (20) is introduced to define a suitable adjoint system.

$$\begin{aligned} & \int_\Omega g_i r \dot{u}_i\, dx + \int_{\partial\Omega} \{T_i(w)\, \dot{u}_i - w_i \dot{T}_i(u)\} r\, ds \\ & = \int_\Omega g_i r \nabla u_i \cdot V\, dx + \int_{\partial\Omega} [\{(f_i w_i - u_{i,s} S_i(w) + T_i(u) \\ & (w_{i,n} + w_i H)) r - u_r \sigma_\theta(w) + T_i(u) w_i n_r\} V_n \\ & + \{u_{i,s} T_i(w) - T_{i,s}(u) w_i\} V_s] ds, \end{aligned} \tag{23}$$

where $[S_i(w) = \sigma_{ij}(w) s_i]$ is the tangential component of the stress tensor on $\partial\Omega$. $\sigma_\theta(w)$ is the stress in the hoop direction, which is given by the stress-strain relation for axisymmetric case:

$$\sigma_\theta(w) = E\varepsilon_\theta + \nu(\sigma_r + \sigma_z), \tag{24}$$

where E and ν are the material constants, σ_r and σ_z are the stresses in r and z direction, respectively, and $\varepsilon_\theta = u_r/r$ is the hoop strain.

To relate $\dot{u}_i$ and $\dot{T}_i(u)$ in (22) and (23) with V, an adjoint system is now defined such that

$$f^*_i = h_{ui} \quad \text{in } \Omega, \tag{25}$$

$$\begin{aligned} u^*_i &= -\Psi_{Ti} && \text{on } \partial\Omega_0, \\ T^*_i &= \Psi_{ui} && \text{on } \partial\Omega_1. \end{aligned} \tag{26}$$

Substituting f^*_i, u^*_i and T^*_i in place of g_i, w_i and $T_i(w)$ in (23), the sensitivity formula for Φ becomes

$$\begin{aligned} \Phi' = {} & \int_{\partial\Omega} [\{(h + \Psi H + f_i u^*_i - u_{i,s} S^*_i + T_i(u^*_{i,n} + u^*_i H)) r \\ & + (\Psi + T_i u^*_i) n_r - u_r \sigma_\theta\} V_n + \{((u_{i,s} T^*_i - T_{i,s} u^*_i) r \\ & - \Psi n_z) V_s + \Psi r V_{s,s}] ds + \int_{\partial\Omega_0} (\Psi_{ui} - T^*_i) r \dot{b}_i\, ds \\ & + \int_{\partial\Omega_1} (\Psi_{Ti} + u^*_i) r \dot{c}_i\, ds. \end{aligned} \tag{27}$$

Note here that S_i, σ_θ and $u_{i,n}$ can expressed by a combination of u_i, T_i and their tangential derivatives:

$$\begin{aligned} S_i &= \tau n_i + \sigma_s s_i, \\ \sigma_\theta &= E\varepsilon_\theta + \nu(\sigma_n + \sigma_s), \\ u_{i,n} &= [\frac{1}{2\mu}\{(1-\nu)\sigma_n - \nu\sigma_s\} - \nu\varepsilon_\theta] n_i + (\frac{1}{\mu}\tau - u_{k,s} n_k) s_i, \end{aligned} \tag{28}$$

where μ is the shear modulus, and σ_n, τ and σ_s are the stress components in (n, s) coordinate, which are given by

$$\begin{aligned} \sigma_n &= T_k n_k, \\ \tau &= T_k s_k, \\ \sigma_s &= \frac{2\mu}{1-\nu}(u_{k,s} s_k + \nu\varepsilon_\theta) + \frac{\nu}{1-\nu}\sigma_n. \end{aligned}$$

4. PLATE BENDING PROBLEM

Consider the problem of finding lateral deflection u of a plate with unit thickness as shown in Fig. 4. The equilibrium equation is given by the biharmonic operator where $2m = 4$ as follows (Timoshenko, 1959).

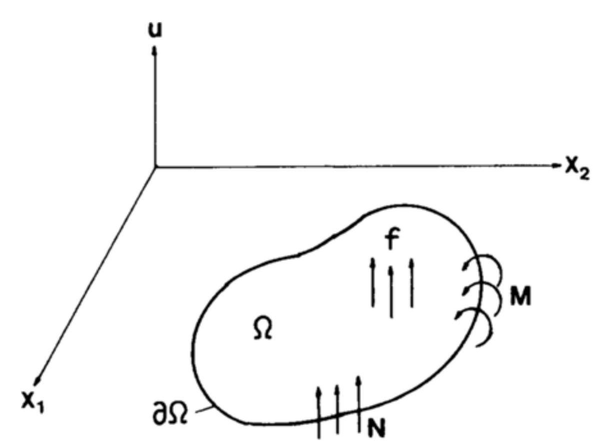

Fig. 4 Plate bending problem

$$Au(x)=\Delta^2 u(x)=\frac{1}{D}f(x), \qquad x\varepsilon\Omega, \tag{29}$$

where D is the bending stiffness given by $E/(1-\nu^2)$, Δ is the Laplacian operator, and $f(x)$ is the applied surface load. The boundary conditions are

$$u=b_0 \quad on \quad \partial\Omega_0, \qquad p(u)\equiv\frac{\partial u}{\partial n}=b_1 \quad on \quad \partial\Omega_1,$$
$$M(u)=c_0, on \quad \partial\Omega_2, \qquad N(u)=c_1 \quad on \quad \partial\Omega_3, \tag{30}$$

where $\partial\Omega_0 \cup \partial\Omega_3=\partial\Omega_1 \cup \partial\Omega_2=\partial\Omega$. The boundary operators B_0, B_1, C_0 and C_1 become in this case 1, $\partial/\partial n$, M and N. M and N, operators which relate the displacement u to the bending moment and equivalent shear force on the boundary are defined as:

$$M(u)=m_{ij}(u)n_i n_j, \tag{31}$$
$$N(u)=Q(u)-T_{,s}(u), \tag{32}$$

The operators m_{ij}, T, & Q are defined by the following:

$$m_{ij}(u)=D\{(1-\nu)u_{,ij}+\nu\delta_{ij}u_{,kk}\}, \tag{33}$$
$$T(u)=m_{ij}(u)n_i s_j, \tag{34}$$
$$Q(u)=-m_{ij,j}(u)n_i. \tag{35}$$

with the assumption of smoothness on the boundary, a pair of BIE's are obtained as follows (Banerjee, 1981),

$$\frac{1}{2}B_i u+\int_{\partial\Omega}\{uN(G_i)+p(u)M(G_i)-M(u)p(G_i) -N(u)G_i\}ds=\int_\Omega fG_i\,dx, \quad x\varepsilon\,\partial\Omega, \; i=0, 1, \tag{36}$$

where the arguments of u and G_i have been omitted for simplicity. G_0 and G_1 denote fundamental solutions with order 0 and 1, respectively. The corresponding integral identity becomes

$$\int_{\partial\Omega}\{uN(w)-p(u)M(w)-M(u)p(w)-N(u)w\}ds =\int_\Omega(fw-gu)\,dx, \tag{37}$$

where w is an arbitrary function satisfying $\Delta^2 w=\frac{1}{D}g$.

Consider now, for the plate bending problem the general functional,

$$\Phi=\int_\Omega h(u, m_{ij})dx+\int_{\partial\Omega}\Psi(u, p, M, N)\,ds. \tag{38}$$

Take the material derivative of Φ, and note that

$$\int_\Omega h_{m_{ij}} m_{ij}(\dot{u})\,dx=\int_{\partial\Omega}\{N^o\dot{u}+M^o\dot{p}+M^o u_{,s}(V_{n,s} - V_s H)\}ds+\int_\Omega m^o{}_{ij,ij}\dot{u}\,dx, \tag{39}$$

where $\quad m^o{}_{ij}=D\{(1-\nu)h_{m_{ij}}+\nu\delta_{ij}h_{m_{kk}}\}$ (40)

and M^o and N^o are given by Eqs. (31) ~ (35).
Then, the material derivative of Φ becomes

$$\Phi'=\int_\Omega(h_u+m^o{}_{ij,ij})\dot{u}\,dx+\int_{\partial\Omega}\{(\Psi_u+N^o)\dot{u}+(\Psi_p +M^o)\dot{p}+\Psi_M\dot{M}+\Psi_N\dot{N}\}ds-\int_\Omega\{h_u\nabla u\cdot V + h_{m_{ij}}m_{ij}(\nabla u\cdot V)\}dx+\int_{\partial\Omega}\{(h+\Psi H)V_n + \Psi V_{s,s}+M^o u_{,s}(V_{n,s}-V_s H)\}ds. \tag{41}$$

As before, the next step is to obtain the material derivative of the identity (37). After taking variations of (37), use the following relations,

$$\dot{p}(w)=p(w')+w_{,nn}V_n+p_{,s}(w)V_s-w_{,s}V_{n,s},$$
$$\dot{M}(w)=M(w')+(m_{ij,n}(w)n_i n_j)V_n+M_{,s}(w)V_s-2T(w)V_{n,s}$$
$$\dot{N}(w)=N(w')-(m_{ij,jn}(w)n_i)V_n+Q_{,s}(w)V_s+D(\Delta w)_{,s}V_{n,s}-\{(m_{ij,n}(w)n_i s_j)V_n+T_{,s}(w)V_s + (M(w)-S(w)V_{n,s}\}_{,s}+T_{,s}(w)(V_n H + V_{s,s}), \tag{42}$$

where S, Eq. (43) denotes the transverse bending moment on $\partial\Omega$.

$$S=m_{ij}S_i S_j, \tag{43}$$

The expressions for the operators β_0, β_1, γ_0 and γ_1(Kwak, 1987a) are included on the righthand side of Eq. (42). After rather lengthy manipulation with various derivatives, the expression for sensitivity of Φ is finally obtained:

$$\Phi'=\int_{\partial\Omega}\{[h+\Psi H+fu^*+u_{,s}(M^*-M^o)_{,s}+p\{(N^* -N^o)-(M^*-M^o)H\}-m_{ij}u^*{}_{,ij}+\Delta u(M^*-M^o) +M\Delta u^*+M_{,s}u^*{}_{,s}+N(p^*+u^*H)]V_n+[u_{,s}(N^* -N^o)+p_{,s}(M^*-M^o)-M_{,s}p^*-N_{,s}u^*]V_s + \Psi V_{s,s}\}ds+\int_{\partial\Omega_0}(\Psi_u+N^o-N^*)\dot{b}_0\,ds+\int_{\partial\Omega_1}(\Psi_p +M^o-M^*)\dot{b}_1\,ds+\int_{\partial\Omega_2}(\Psi_M+p^*)\dot{c}_0\,ds+\int_{\partial\Omega_3}(\Psi_N + u^*)\dot{c}_1\,ds, \tag{44}$$

where the adjoint system is defined such that

$$f^*=h_u+m^o{}_{ij,ij} \qquad in\ \Omega, \tag{45}$$
$$u^*=-\Psi_N \quad on\ \partial\Omega_0, \qquad p^*=-\Psi_M \quad on\ \partial\Omega_1$$
$$M^*=\Psi_p+M^o \quad on\ \partial\Omega_2, \qquad N^*=\Psi_u+N^o \quad on\ \partial\Omega_3 \tag{46}$$

A plate bending problem under the clamped boundary conditions, i.e., $u=0$, $p=0$ on $\partial\Omega$, is used for specialization of the derived formula. A compliance functional for the plate is considered:

$$\Phi=\int_\Omega fu\,dx. \tag{47}$$

Noting from (45) and (46) that the adjoint variable u^* is identical to u, the lengthy sensitivity Eq. (44) becomes simply

$$\Phi'=\int_{\partial\Omega}\{-m_{ij}u_{,ij}+2\Delta uM\}V_n\,ds. \tag{48}$$

Because the tangential derivatives of u and p are zero on $\partial\Omega$, Eq. (48) is further simplified to

$$\Phi'=D\int_{\partial\Omega}(u_{,nn})^2\,V_n\,ds. \tag{49}$$

The same reulst is obtained, when a variational approach is used (Choi, 1983).

5. CONCLUSION AND DISCUSSIONS

Based upon previously derived general formulas, concrete formulas for shape sensitivity related to the three typical engineering problems of potential, axisymmetric elasticity and plate bending design are derived. These formulas are specific enough for computational purposes. The calculation of these formulas is dependent upon the primary BIE only in the prescribed data on the boundary and domain, the same BEM equations with different input data can be used. Numerical implementations of the formulas are being studied as well as the treatment of discontinuities in boundaries and those related functions.

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